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Approximate Symmetries and Conservation Laws with Applications

A. H. Kara,[1] F. M. Mahomed,[2,4] and G. Ünal[3]

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The relationship between the approximate Lie–Bäcklund symmetries and the approximate conserved forms of a perturbed equation is studied. It is shown that a hierarchy of identities exists by which the components of the approximate conserved vector or the associated approximate Lie–Bäcklund symmetries are determined by recursive formulas. The results are applied to certain classes of linear and nonlinear wave equations as well as a perturbed Korteweg–de Vries equation. We construct approximate conservation laws for these equations without regard to a Lagrangian.

1. INTRODUCTION

For differential equations, the relationship between symmetries and conservation laws has been a subject of intensive investigation, the best known being the work of Noether (1918) for Euler–Lagrange equations (see also Ibragimov *et al.* 1998). Recently Kara and Mahomed (1998, 1999) showed that for differential equations a formula governs the relation between the components of the conserved vector and the *associated* Lie–Bäcklund symmetry generator for systems which need not be derivable from a variational principle, for, e.g., evolution-type equations. This was used mainly to construct conservation laws for such systems.

[1] Department of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa; e-mail: 036ahk@cosmos.wits.ac.za.
[2] Department of Computational and Applied Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa; e-mail:fmahomed@cam.wits.ac.za.
[3] Faculty of Science and Letters, Istanbul Technical University, Maslak 80626, Istanbul, Turkey; e-mail:gunal@itu.edu.tr.
[4] To whom correspondence should be addressed.

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The theory of approximate symmetries associated with differential equations with a small parameter (perturbed/approximate equations) has been developed recently and is a subject of much discussion. For an account of this theory, we refer the reader to Baikov *et al.* (1991, 1996). In Baikov *et al.* (1996), the notion of approximate conservation laws is introduced with specific regard to approximate Noether symmetries, i.e., symmetries associated with a Lagrangian of the perturbed equation. In this paper, we consider these ideas without recourse to variational principles, i.e., we show that approximate conservation laws can be associated with approximate symmetries (more generally, Lie–Bäcklund symmetries) for systems which may not possess a Lagrangian. Moreover, we show that these conservation laws may be constructed algorithmically using known symmetries of the equations. One can thus automate this procedure using computer algebra.

Here we consider differential equations of the form

$$E^{\beta}(x, u, u_{(1)}, \ldots, u_{(r)}; \epsilon) = 0, \qquad \beta = 1, \ldots, \tilde{m} \tag{1.1}$$

where $x = (x^1, x^2, \ldots, x^n)$, $u = (u^1, u^2, \ldots, u^m)$, ϵ is a small parameter, and $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ are various order derivatives, namely $u_i^{\alpha} = D_i(u^{\alpha})$, $u_{ij}^{\alpha} = D_jD_i(u^{\alpha})$, ..., being the first, second, ..., derivatives, respectively, with

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha}\frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha}\frac{\partial}{\partial u_j^{\alpha}} + \cdots, \qquad i = 1, \ldots, n \tag{1.2}$$

the total derivative operator with respect to x^i. Throughout the paper, the following definitions of approximate symmetries and conservation laws of (1.1) will be adopted. For more details, see Baikov *et al.* (1991, 1996).

Definition 1. An operator χ is a kth-order approximate symmetry of (1.1) if

$$\chi(E^{\beta})\big|_{E^{\beta}=0} = O(\epsilon^{k+1}) \tag{1.3}$$

where

$$\chi = X_0 + \epsilon X_1 + \cdots + \epsilon^k X_k \tag{1.4}$$

and

$$X_b = \xi_b^i\frac{\partial}{\partial x^i} + \eta_b^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \zeta_{b,i}^{\alpha}\frac{\partial}{\partial u_i^{\alpha}} + \zeta_{b,i_1i_2}^{\alpha}\frac{\partial}{\partial u_{i_1i_2}^{\alpha}} + \cdots, \qquad b = 1, \ldots, k \tag{1.5}$$

where ξ_b^i, $\eta_b^\alpha \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned}\zeta_{b,i}^\alpha &= D_i(W_b^\alpha) + \xi_b^j u_{ij}^\alpha \\ \zeta_{b,i_1i_2}^\alpha &= D_{i_1}D_{i_2}(W_b^\alpha) + \xi_b^j u_{ji_1i_2}^\alpha \\ &\ldots\end{aligned} \tag{1.6}$$

and W_b^α is the Lie characteristic function defined by

$$W_b^\alpha = \eta_b^\alpha - \xi_b^j u_j^\alpha \tag{1.7}$$

Note. The X_b are Lie–Bäcklund operators. In Baikov *et al.* (1991, 1996), the case $X_0 \neq 0$ defines X_0 as a *stable* symmetry and *unstable* otherwise.

Definition 2. An approximate conserved vector $\mathcal{T} = (\mathcal{T}^1, \mathcal{T}^2, \ldots, \mathcal{T}^n)$ of (1.1) satisfies

$$D_i\mathcal{T}^i\Big|_{(1.1)} = O(\epsilon^{k+1}) \tag{1.8}$$

where

$$\mathcal{T}^i = T_0^i + \epsilon T_1^i + \ldots + \epsilon^k T_k^i \tag{1.9}$$

Equation (1.8) is an approximate conservation law for (1.1).

We can define stable and unstable conservation laws in accordance with the above.

In Section 2, we present the relationship between the approximate conservation laws and approximate symmetries for the system (1.1). In Section 3 we demonstrate our results by considering some examples that arise in mathematical physics.

2. APPROXIMATE SYMMETRIES AND CONSERVATION LAWS

In this section, we investigate and extend the ideas and results in Kara and Mahomed (1998, 1999) to the perturbed differential equation (1.1).

Definition 3. A p-form

$$\omega = f_{i_1i_2\ldots i_p}(x, u, u_{(1)}, \ldots, u_{(l)}; \epsilon)\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \tag{2.1}$$

is approximately conserved (of order k) if

$$\mathrm{D}\omega\Big|_{(1.1)} = O(\epsilon^{k+1}) \tag{2.2}$$

where

$$\mathrm{D}\omega = (D_j f_{i_1i_2\ldots i_p})\, dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p}$$

Lemma 1. The $(n-1)$-form

$$\omega = \mathcal{T}^i \frac{\partial}{\partial x^i} \rfloor \Omega \tag{2.3}$$

where $\Omega = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, is approximately conserved if and only if $D_i \mathcal{T}^i \big|_{(1.1)} = O(\epsilon^{k+1})$.

Proof. This follows directly since

$$\mathrm{D}\omega = (D_i \mathcal{T}^i)\Omega = O(\epsilon^{k+1})\Omega$$

Definition 4. A p-form (2.1) is approximately invariant under χ if

$$\mathcal{L}_\chi \omega = O(\epsilon^{k+1}) \tag{2.4}$$

where

$$\mathcal{L}_X \omega = X \rfloor \mathrm{D}\omega + \mathrm{D}(X \rfloor \omega) \tag{2.5}$$

i.e., the Cartan formula is valid.

Theorem 1. The $(n-1)$-form given in (2.3) is approximately invariant under χ if and only if the following system of $n(k+1)$ identities hold:

$$\begin{aligned}
&X_0(T^i_0) + D_j(\xi^j_0)T^i_0 - T^j_0\, D_j(\xi^i_0) = 0 \\
&X_0(T^i_1) + D_j(\xi^j_0)T^i_1 - T^j_1 D_j(\xi^i_0) = -(X_1(T^i_0) + D_j(\xi^j_1)T^i_0 - T^j_0 D_j(\xi^i_1)) \\
&X_0(T^i_2) + D_j(\xi^j_0)T^i_2 - T^j_2 D_j(\xi^i_0) = -(X_2(T^i_0) + D_j(\xi^j_2)T^i_0 - T^j_0 D_j(\xi^i_2) \\
&\qquad\qquad + X_1(T^i_1) + D_j(\xi^j_1)T^i_1 - T^j_1 D_j(\xi^i_1)) \qquad (2.6) \\
&\qquad\vdots \\
&X_0(T^i_k) + D_j(\xi^j_0)T^i_k - T^j_k D_j(\xi^i_0) = -(X_k(T^i_0) + D_j(\xi^j_k)T^i_0 - T^j_0 D_j(\xi^i_k) \\
&\qquad\qquad + \ldots + X_1(T^i_{k-1}) + D_j(\xi^j_1)T^i_{k-1} \\
&\qquad\qquad - T^j_{k-1} D_j(\xi^i_1))
\end{aligned}$$

Proof. The kth-order approximate form ω reads

$$\omega = \omega_0 + \epsilon\omega_1 + \ldots + \epsilon^k \omega_k \tag{2.7}$$

Then (2.4) takes the form

$$\mathcal{L}_{X_0 + \epsilon X_1 + \ldots + \epsilon^k X_k}(\omega_0 + \epsilon\omega_1 + \ldots + \epsilon^k \omega_k) = O(\epsilon^{k+1}) \tag{2.8}$$

Rearranging (2.8) gives

$$\mathcal{L}_{X_0}\omega_0 + \epsilon(\mathcal{L}_{X_1}\omega_0 + \mathcal{L}_{X_0}\omega_1) + \epsilon^2(\mathcal{L}_{X_2}\omega_0 + \mathcal{L}_{X_1}\omega_1 + \mathcal{L}_{X_0}\omega_2) + \dots$$
$$+ \epsilon^k(\mathcal{L}_{X_k}\omega_0 + \mathcal{L}_{X_{k-1}}\omega_1 + \dots + \mathcal{L}_{X_0}\omega_k) = 0$$

Separating by powers of ϵ, we obtain

$$\begin{aligned} \mathcal{L}_{X_0}\omega_0 &= 0 \\ \mathcal{L}_{X_1}\omega_0 + \mathcal{L}_{X_0}\omega_1 &= 0 \\ &\vdots \\ \mathcal{L}_{X_k}\omega_0 + \mathcal{L}_{X_{k-1}}\omega_1 + \dots + \mathcal{L}_{X_1}\omega_{k-1} + \mathcal{L}_{X_0}\omega_k &= 0 \end{aligned} \tag{2.9}$$

Now let $\mathcal{T}_b = T^j_b\, \partial/\partial x^j$, $b = 0, \dots, k$. Then

$$\omega_s = \mathcal{T}_s \rfloor \Omega$$

In the identity

$$\mathcal{L}_{X_r}(\mathcal{T}_s \rfloor \Omega) = [X_r, \mathcal{T}_s] \rfloor \Omega + \mathcal{T}_s \rfloor \mathcal{L}_{X_r}\Omega \tag{2.10}$$

we have

$$\mathcal{L}_{X_r}\Omega = (D_j\xi^j_r)\Omega \tag{2.11}$$

and

$$[X_r, \mathcal{T}_s] = (X_r(T^i_s) - D_j(\xi^i_r)T^j_s)\frac{\partial}{\partial x^i} \tag{2.12}$$

so that

$$\mathcal{L}_{X_r}(\mathcal{T}_s \rfloor \Omega) = \left\{(X_r(T^i_s) - D_j(\xi^i_r)T^j_s + D_j\xi^j_r)\frac{\partial}{\partial x^i}\right\} \rfloor \Omega \tag{2.13}$$

Substituting equation (2.13) into (2.9) yields the required result in (2.6).

As (2.6) is connected to (2.9) via the identity in (2.10), the steps given above are easily reversed. This proves the converse.

Remarks. Special cases of (2.6) are as follows.

1. If each of the X_b in χ are canonical, the system (2.6) becomes

$$\begin{aligned} X_0(T^i_0) &= 0 \\ X_0(T^i_1) &= -X_1(T^i_0) \\ X_0(T^i_2) &= -X_2(T^i_0) - X_1(T^i_1) \\ &\vdots \end{aligned}$$

$$X_0(T^i_k) = -X_k(T^i_0) - \ldots - X_1(T^i_{k-1})$$

2. For the case of a single independent variable we have the following system of $(k + 1)$ ordinary differential equations

$$\begin{aligned} X_0(T_0) &= 0 \\ X_0(T_1) &= -X_1(T_0) \\ X_0(T_2) &= -X_2(T_0) - X_1(T_1) \\ &\vdots \\ X_0(T_k) &= - X_k(T_0) - \ldots - X_1(T_{k-1}) \end{aligned}$$

Corollary. The $(n - 2)$-form

$$\bar{\omega} = \chi \rfloor \left(\mathcal{T}^i \frac{\partial}{\partial x^i} \right) \rfloor \Omega \tag{2.14}$$

is approximately conserved.

Proof. From the identity (2.10), we can write

$$\begin{aligned} D\left(\chi \rfloor \mathcal{T}^i \frac{\partial}{\partial x^i} \rfloor \Omega \right) &= \mathcal{L}_\chi \left(\mathcal{T}^i \frac{\partial}{\partial x^i} \rfloor \Omega \right) - \chi \rfloor D\left(\mathcal{T}^i \frac{\partial}{\partial x^i} \rfloor \Omega \right) \\ &= O(\epsilon^{k+1}) \end{aligned} \tag{2.15}$$

The proof follows directly from Lemma 1 and Theorem 1.

3. APPLICATIONS

We illustrate our results by considering some examples from the literature. We construct approximate conservation laws for examples where they do exist and we also give examples when they do not exist.

It is interesting to note that a conservation law of a given equation need not be stable even though its approximate conservation law is constructed from a stable symmetry. That is, if one starts from a stable symmetry X_0 with associated exact conserved vector (T^1_0, T^2_0) of the unperturbed or exact equation, one may not end up with an approximate conservation law associated with the approximate symmetry $\chi = X_0 + \epsilon X_1 + \ldots$ of the perturbed equation. We give examples of these. On the other hand, a symmetry need not have to be stable to give rise to an approximate conservation law of the perturbed equation. This, too, is illustrated below.

Example 1. It is easy to verify that the linear perturbed wave equation

$$u_{tt} - u_{xx} + \epsilon u_t = 0 \tag{3.1}$$

admits the approximate symmetry $\chi = X_0 + \epsilon X_1$, where $X_0 = \partial/\partial t$ and $X_1 = -\frac{1}{2}u\,\partial/\partial u$. It is clear that the unperturbed equation has conserved vector with components $T_0^1 = u_t$ and $T_0^2 = -u_x$ and has associated symmetry X_0.

Here the second set of equations of (2.6), viz. $X_0T_1^1 = -X_1T_0^1$ and $X_0T_1^2 = -X_1T_0^2$, becomes

$$\frac{\partial}{\partial t}T_1^1 = \frac{1}{2}u_t \qquad \text{and} \qquad \frac{\partial}{\partial t}T_1^2 = -\frac{1}{2}u_x$$

so that $T_1^1 = \frac{1}{2}tu_t + f(x, u, u_t, u_x)$ and $T_1^2 = -\frac{1}{2}tu_x + g\,(x, u, u_t, u_x)$. Substituting into the conserved form of the perturbed case, viz.,

$$D_tT_1^1 + D_xT_1^2 = u_t$$

yields, after setting $g = 0$,

$$T_1^1 = \frac{1}{2}tu_t + \frac{1}{2}u, \qquad T_1^2 = -\frac{1}{2}tu_x$$

It is simple to check that

$$(\mathcal{T}^1, \mathcal{T}^2) = (T_0^1 + \epsilon T_1^1, T_0^2 + \epsilon T_1^2) = \left(u_t + \epsilon\left[\frac{1}{2}tu_t + \frac{1}{2}u\right], -u_x - \epsilon\frac{1}{2}tu_x\right)$$

is a first-order approximate conserved vector of (3.1) since (1.8) for $\mathcal{T}^1$ and $\mathcal{T}^2$ gives

$$u_{tt} - u_{xx} + \epsilon u_t = -\frac{1}{2}\epsilon^2 tu_t$$

It is interesting to note that another conserved vector of the unperturbed equation (3.1) has components $T_0^1 = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2$ (which gives rise to conservation of energy) and $T_0^2 = -u_tu_x$ which have the same associated symmetry, $X_0 = \partial/\partial t$, as above. Here, $D_tT_0^1 + D_xT_0^2 = u_t(u_{tt} - u_{xx})$. Then, $D_tT_1^1 + D_xT_1^2 = u_t^2$ is the conserved form to be solved simultaneously with (2.6). The calculations give

$$T_1^1 = \frac{1}{2}tu_t^2 + \frac{1}{2}tu_x^2 + \frac{1}{2}uu_t - \frac{1}{2}uu_x$$

$$T_1^2 = -u_tu_xt + \frac{1}{2}uu_t - \frac{1}{2}uu_x$$

As a third case, the conserved vector with components $T_0^1 = u_t$ and $T_0^2 = -u_x$ is associated with the symmetry $\partial/\partial u$. A corresponding approximate symmetry of (3.1) is $\partial/\partial u + \epsilon u\, \partial/\partial u$. Here the calculations reveal that no first-order approximate conservation law exists.

As a nonlinear example, consider the wave equation

$$u_{tt} + \epsilon u u_t = u_{xx}$$

For this equation, an approximate symmetry is (Baikov *et al.*, 1991) $\chi = X_0 + \epsilon X_1$, where $X_0 = \partial/\partial u$ is stable and $X_1 = -\frac{1}{2} tu\, \partial/\partial u$. The routine calculations for both pairs of (T_0^1, T_0^2) given above, viz. $(u_t, -u_x)$ and $(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2, -u_t u_x)$, both of which have associated symmetry $\partial/\partial u$, once again provide no approximate first-order associated pairs (T_1^1, T_1^2) and hence no approximate first-order conservation law for these cases.

Remark. It is straightforward and simple to observe that the wave equation

$$u_{tt} + \epsilon f'(u) u_t = u_{xx}$$

has the conserved vector

$$(\mathcal{T}^1, \mathcal{T}^2) = (u_t + \epsilon f(u), -u_x)$$

which has associated symmetry $\chi = \partial/\partial t + \epsilon\, \partial/\partial x$ or $\chi = \partial/\partial x + \epsilon\, \partial/\partial t$. These cases correspond to the exact symmetry and exact conservation law of the equation.

Example 2. The nonlinear wave equation

$$u_{tt} + \epsilon u_t = h(u_x) u_{xx} \tag{3.2}$$

has been analyzed in Baikov *et al.* (1996). The conservation laws for the exact equation ($\epsilon = 0$) for the case $h = u_x^\alpha$, $\alpha \neq 0, -1, -2$, is discussed in Vinokurov and Nurgalieva (1985). We first consider the situation when no first-order approximate conservation law exists with respect to the approximate symmetry $\chi = (t - \frac{1}{2}\epsilon t^2)\, \partial/\partial u$ admitted by equation (3.2). The exact conserved vector (T_0^1, T_0^2) with associated symmetry $t\, \partial/\partial u$, which is stable, is obtained from

$$t \frac{\partial T_0^1}{\partial u} + \frac{\partial T_0^1}{\partial u_t} = 0, \qquad t \frac{\partial T_0^2}{\partial u} + \frac{\partial T_0^2}{\partial u_t} = 0$$

which yields

$$T_0^1 = \beta(x)\, t u_x + \gamma(x)(u - t u_t) + \delta(x)$$

$$T_0^2 = \beta(x(u - t u_t) + \mu(x) + t\gamma(x) \int h(u_x)\, du_x + \nu(t)$$

Substituting these into the corresponding conserved law $D_tT_0^1 + D_xT_0^2 = 0$ gives rise to $T_0^1 = tu_t - u$, $T_0^2 = -t \int h(u_x)\, du_x$. Thus, the equation

$$u_{tt} + \epsilon u_t = u_x^\alpha u_{xx}, \qquad \alpha \neq -1 \tag{3.3}$$

with $T_0^1 = tu_t - u$, $T_0^2 = -[1/(\alpha + 1)]\, tu_x^{\alpha+1}$, has approximate conserved vector $(T_0^1 + \epsilon T_1^1, T_0^2 + \epsilon T_1^2)$ associated with $\chi = X_0 + \epsilon X_1 = (t - \frac{1}{2}\epsilon t^2)\, \partial/\partial u$, where T_1^1 and T_1^2 are deduced from (2.6):

$$X_0T_1^1 = -X_1T_0^1, \qquad X_0T_1^2 = -\, X_1T_0^2$$

We have that $T_1^1 = -\frac{1}{2}tu + f(t, x, u_t, u_x)$ and $T_1^2 = g(t, x, u_t, u_x)$. The corresponding conserved form

$$D_tT_1^1 + D_xT_1^2 = tu_t$$

gives incompatibilities, from which we conclude that (3.3) has no first-order approximate conservation law associated with the given χ.

However, a similar analysis carried out on (3.2) with $h = u_x^\alpha$, $\alpha \neq -1, -2$, with the approximate symmetry $X = \partial/\partial t + \epsilon\, \partial/\partial u$ yields

$$T_0^1 = -\tfrac{1}{2}u_t^2 - \frac{1}{(\alpha + 1)(\alpha + 2)}\, u_x^{\alpha+2}, \qquad \alpha \neq -1, -2$$

$$T_0^2 = \frac{1}{\alpha + 1}\, u_tu_x^{\alpha+1}$$

Solving $X_0T_1^1 = -X_1T_0^1$, $X_0T_1^2 = -X_1T_0^2$ together with the corresponding conserved form $D_tT_1^1 + D_xT_1^2 = -u_t^2$ yields the approximate conserved vector components

$$T_1^1 = 2\,\frac{\alpha + 2}{3\alpha + 4}\, xu_xu_t + \frac{\alpha + 2}{3\alpha + 4}\, uu_t - uu_t$$

$$T_1^2 = -\frac{\alpha + 2}{3\alpha + 4}\, xu_t^2 + \frac{1}{\alpha + 1}\, uu_x^{\alpha+1}$$

$$- \frac{\alpha + 2}{(\alpha + 1)(3\alpha + 4)}\, uu_x^{\alpha+1} - \frac{2}{3\alpha + 4}\, xu_x^{\alpha+2}$$

provided $\alpha \neq -1, -2, -4/3$ and hence a first-order approximate conservation law for (3.3).

Example 3. We now look at the perturbed nonlinear wave equation

$$u_{tt} + \epsilon\left(uu_t + \frac{1}{2}\, tu_t^2 - \frac{1}{2}\, tu_x^2 \right) = u_{xx} \tag{3.4}$$

which has approximate symmetry $\chi = X_0 + \epsilon X_1$ with $X_0 = \partial/\partial u$ stable [X_0

has associated exact conserved vector $(T_0^1, T_0^2) = (u_t, -u_x)]$ and $X_1 = -\frac{1}{2}tu\,\partial/\partial u$. The conditions for an approximate conservation law with symmetry χ result in

$$\frac{\partial T_1^1}{\partial u} = \frac{1}{2}u + \frac{1}{2}tu_t, \qquad \frac{\partial T_1^2}{\partial u} = -\frac{1}{2}tu_x$$

The simplest solution is

$$T_1^1 = \frac{1}{4}u^2 + \frac{1}{2}tuu_t, \qquad T_1^2 = -\frac{1}{2}tuu_x$$

and hence an approximate first-order conservation law results.

We now consider a third-order evolution equation.

Example 4. A number of particular perturbed cases of the Korteweg–de Vries equation

$$u_t = uu_x + u_{xxx} + \epsilon f(u, u_x, u_{xx}, \ldots) \tag{3.5}$$

have been considered in Baikov *et al.* (1991, 1996). In this example, we look at two cases. First we search for an f for which (3.5) admits an approximate conservation law corresponding to the approximate symmetry $t\,\partial/\partial x - \partial/\partial u + \epsilon\,\partial/\partial t$.

An approximate symmetry of (3.5) is $\chi = X_0 + \epsilon X_1 = t\,\partial/\partial x - \partial/\partial u + \epsilon\,\partial/\partial t$. A conserved vector associated with the exact symmetry is $(T_0^1, T_0^2) = (xu_x + tuu_x, -xu_t - tuu_t + u_{xx})$. Let us now invoke the approximate symmetry condition for the conserved vector, viz. (2.6), which gives $X_0T_1^1 = -X_1T_0^1$ and $X_0T_1^2 = -X_1T_0^2$. The solutions for T_1^1 and T_1^2 are

$$T_1^1 = \frac{1}{2}u^2u_x + g, \qquad T_1^2 = -\frac{1}{2}u^2u_x + \frac{1}{2}u^2(uu_x - u_t) + h$$

where g and h are as yet arbitrary functions of t, u_x, u_{tx}, u_{xx}, $x + ut$, $u_xu - u_t$. Now we impose the condition for approximate conservation law $D_tT_1^1 + D_x T_1^2 = f$. An admissible function f, for $g = 0$ and $h = 0$, is $f = f_1 = u_{xx}(u^3/2 - u^2/2) + u_x^2(3u^2/2 - u)$. Hence an approximate conserved vector of (3.5) with $f = f_1$ corresponding to the approximate symmetry $t\partial/\partial x - \partial/\partial u + \epsilon\,\partial/\partial t$ is

$$(\mathcal{T}^1, \mathcal{T}^2) = \left(xu_x + tuu_x + \epsilon\frac{1}{2}u^2\,u_x,\ -xu_t - tuu_t + u_{xx} + \epsilon\frac{1}{2}u^2\,[-u_x - u_t + uu_x] \right)$$

Now we show that equation (3.5), for $f = u$, with respect to the approximate symmetry

$$\partial/\partial t - c\,\partial/\partial x - 3\epsilon t\,\partial/\partial t + \epsilon(ct - x)\,\partial/\partial x + \epsilon(2u - c)\,\partial/\partial u$$

does not admit a second-order approximate conservation law.

An approximate symmetry admitted by (3.5) for $f = u$ is $\chi = X_0 + \epsilon X_1$, where

$$X_0 = \partial/\partial t - c\,\partial/\partial x, \qquad X_1 = -3t\,\partial/\partial t + (ct - x)\,\partial/\partial x + (2u - c)\,\partial/\partial u$$

A conserved vector associated with X_0 has components $T_0^1 = \frac{1}{2}u^2$ and $T_0^2 = \frac{1}{2}u_x^2 - uu_{xx} - \frac{1}{3}u^3$. Now the relations (2.6), viz. $X_0T_1^1 = -X_1T_0^1 + T_0^1$ and $X_0T_1^2 = -X_1T_0^2 + 3T_0^2 + cT_0^1$, yield

$$T_1^1 = (cu - \tfrac{3}{2}u^2)t + g$$

$$T_1^2 = (cu_{xx} - 3u^3 + \tfrac{3}{2}cu^2 + \tfrac{9}{2}u_x^2 - 9uu_{xx})t + h$$

where g and h are functions of u, $x + ct$, u_t, u_x, u_{tt}, u_{tx}, u_{xx}. Substituting these into the conserved of the approximate part gives $D_tT_1^1 + D_xT_1^2 = -u^2$, from which, after straightforward but tedious calculations, it is determined that no solution for T_1^1 and T_1^2 exists, i.e., the approximate symmetry χ of (3.5) for $f = u$ has no associated approximate second-order conservation law.

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